Sequences, Series, and Power Series

Example: Consider the sequence $a_n = \frac{2n+1}{3n}$

$$\lim_{n\to\infty} \frac{2n+1}{3n} = \frac{2}{3}$$

1. The series formed from this sequence is

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n+1}{3n} = 1 + \frac{5}{6} + \frac{7}{9} + \cdots$$

diverges to ∞.

2. The alternating series formed from this sequence is

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{3n} = 1 - \frac{5}{6} + \frac{7}{9} - \cdots$$

diverges by the alternating series test

3. A power series formed from this sequence is

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \left(\frac{2n+1}{3n}\right) x^n = x + \frac{5x^2}{6} + \frac{7x^3}{9} + \cdots$$

Then the radius of convergence and the interval of convergence are found by determining:

$$\ell = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{2n+3}{3n+3}\right)}{\left(\frac{2n+1}{3n}\right)}\right| = \lim_{n\to\infty} \left(\frac{2n+3}{3n+3}\right)\left(\frac{3n}{2n+1}\right) = 1$$

by definition, the radius of convergence R is $R = \frac{1}{\ell} = 1$. We know that the power series converges for $-1 < x < 1$ since

$$\lim_{n\to\infty} \left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = |x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x| < 1$$ to converge by the ratio test.

and diverges on $(-\infty,-1)\cup(1,\infty)$. At x=1 or x=-1, we must check the resulting series. At x=1, we get $\sum_{n=1}^{\infty} \frac{2n+1}{3n}$ which we already saw diverges, and at x=-1, we get $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{3n}$ which we already saw diverges.

Thus, the interval of convergence is (-1,1)